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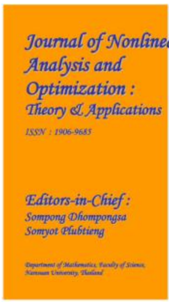

ON CERTAIN TYPE OF SUMMATION METHOD

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Abstract

Summability Theory which began as an independent study few centuries ago, is a part of Number Theory and Mathematical Analysis. It is an alternative formulation of obtaining convergence of an infinite series which is divergent in the conventional sense. In this paper, we consider infinite sum of the k^{th} power of first n natural numbers which is a divergent series and when trying to express in terms of a particular integral taken over bounded interval, we obtain nice results. We will prove such results in this paper. We have provided the verification of such results.

Keywords: Sum of k^{th} Powers, Divergent Series, Bounded Interval, Riemann Integral.

1. Introduction

The notion of convergence was instrumental to the development of various summability methods. Later on, the concept of the absolute summability was developed from the notion of absolute convergence. There are various methods due to Abel, Borel, Cesaro. Euler, Norlund, Riemann, Riesz and many more. During early years of 20^{th} century, various methods of associating sums with series which are neither convergent nor properly divergent were developed as generalization of classical concepts of convergence. These methods called "summability methods" have been found incredibly useful in the study of divergent series. Pioneer researchers such as Holder, Cesaro, Haudesoff, Borel did spectacular work so that summability theory became independently well established and considered as part of modern mathematical analysis. In this paper, we will develop one such summability method and derive nice results depending upon it.

2. Notations and Definitions

1. $$S_k(n) = \sum_{n=1}^{\infty} n^k = 1^k + 2^k + 3^k + 4^k + \cdots \qquad (1)$$

$S_k(n)$ is the sum of the k^{th} powers of positive integers.

2. $$t_k(m) = \sum_{n=1}^{m} n^k = 1^k + 2^k + 3^k + 4^k + \cdots + m^k \qquad (2)$$

$t_k(m)$ is the m^{th} partial sum of $S_k(n)$.

Definition

For $\alpha > 0$ we define

$$S_{k,\alpha}(n) = \int_{m=a}^{b} t_k(m)\, dm \qquad (3)$$

where $[a, b]$ is any interval in R.

If the integral in RHS of (3) exists, then we say that $S_{k,\alpha}(n)$ is Summable.

3. Theorem 1

If k is an even positive integer, and if $S_{k,\alpha}(n) = \int_{m=-\alpha}^{\alpha} t_k(m)\, dm$ then $S_{k,\alpha}(n) = \frac{\alpha^{k+1}}{k+1}$ (4), where α > 0 is a finite real number.

Proof:

From the definition,

$$S_{k,\alpha}(n) = \int_{-\alpha}^{\alpha} t_k(m)\, dm$$

By Faulhaber's Formula (see **[1]**) which states that

$$1^k + 2^k + \cdots + m^k = \frac{m^{k+1}}{k+1} + \frac{m^k}{2} + k\frac{m^{k-1}}{12} + C_{k-3}m^{k-3} + C_{k-5}m^{k-5} + \cdots + C_1 m \quad (5)$$

where C_1 is Bernoulli number and C_i's are constants.

In view of Faulhaber's formula, since $t_k(m)$ is a polynomial in m of degree k + 1, we notice that $\int_{m=-\alpha}^{\alpha} t_k(m)\, dm$ is integrable for all values of α, since $[-\alpha, \alpha]$ is bounded.

From (2), we get,

$$S_{k,\alpha}(n) = \int_{-\alpha}^{\alpha}(1^k + 2^k + \cdots .. + m^k)\, dm \quad (6)$$

Substituting (5) in (6) we get

$$= \int_{-\alpha}^{\alpha}\left(\frac{m^{k+1}}{k+1} + \frac{m^k}{2} + k\frac{m^{k-1}}{12} + C_{k-3}m^{k-3} + C_{k-5}m^{k-5} + \cdots + C_1 m\right) dm$$

$= \int_{-\alpha}^{\alpha}\left(\frac{m^k}{2}\right) dm$ (since k is even and $\int_{-\alpha}^{\alpha} f(x)\, dx = 0$ *for odd functions* f)

$$= \left[\frac{m^{k+1}}{2(k+1)}\right]_{-\alpha}^{\alpha}$$

$$= 2 \cdot \frac{1}{2}\left[\frac{m^{k+1}}{k+1}\right]_0^{\alpha}$$

$$= \frac{\alpha^{k+1}}{k+1}$$

This completes the proof.

Verification of Result (4)

The above theorem can be verified for k = 2, 4, 6, 8,10

When k = 2, $t_2(m) = 1^2 + 2^2 + 3^2 + 4^2 + \cdots + m^2 = \frac{2m^3+3m^2+m}{6}$

Substituting k = 2 in (5)

$$S_{2,\alpha}(n) = \int_{-\alpha}^{\alpha} t_2(m)\, dm = \int_{-\alpha}^{\alpha}\left(\frac{2m^3+3m^2+m}{6}\right) dm = \frac{\alpha^3}{3}$$

When k = 4, $t_4(m) = 1^4 + 2^4 + 3^4 + 4^4 + \cdots + m^4 = \left(\frac{m^5}{5} + \frac{m^4}{2} + \frac{m^3}{3} - \frac{m}{30}\right)$ Substituting k = 4 in (5)

$$S_{4,\alpha}(n) = \int_{-\alpha}^{\alpha} t_4(m)\, dm = \int_{-\alpha}^{\alpha}\left(\frac{m^5}{5} + \frac{m^4}{2} + \frac{m^3}{3} - \frac{m}{30}\right) dm = \frac{\alpha^5}{5}$$

When k = 6, $t_6(m) = 1^6 + 2^6 + 3^6 + 4^6 + \cdots + m^6 = \frac{m^7}{7} + \frac{m^6}{2} + \frac{m^5}{2} - \frac{m^3}{6} + \frac{m}{42}$

Substituting k = 6 in (5)

$$S_{6,\alpha}(n) = \int_{-\alpha}^{\alpha} t_6(m)\, dm = \int_{-\alpha}^{\alpha}\left(\frac{m^7}{7} + \frac{m^6}{2} + \frac{m^5}{2} - \frac{m^3}{6} + \frac{m}{42}\right) dm = \frac{\alpha^7}{7}$$

When k = 8,

$$t_8(m) = 1^8 + 2^8 + 3^8 + 4^8 \ldots + m^8 = \frac{m^9}{9} + \frac{m^8}{2} + \frac{2m^7}{3} - \frac{7m^5}{15} + \frac{2m^3}{9} - \frac{1}{30}m$$

Substituting k = 8 in (5)

$$S_{8,\alpha}(n) = \int_{-\alpha}^{\alpha} t_8(m)\, dm = \int_{-\alpha}^{\alpha}\left(\frac{m^9}{9} + \frac{m^8}{2} + \frac{2m^7}{3} - \frac{7m^5}{15} + \frac{2m^3}{9} - \frac{1}{30}m\right) dm = \frac{\alpha^9}{9}$$

When k = 10,

$$t_{10}(m) = 1^{10} + 2^{10} + 3^{10} \ldots + m^{10} = \frac{m^{11}}{11} + \frac{m^{10}}{2} + \frac{5m^9}{6} - m^7 + m^5 - \frac{1}{2}m^3 + \frac{5}{66}m$$

Substituting $k = 10$ in (5)

$$S_{10,\alpha}(n) = \int_{-\alpha}^{\alpha} t_{10}(m)\, dm$$

$$= \int_{-\alpha}^{\alpha} \left(\frac{m^{11}}{11} + \frac{m^{10}}{2} + \frac{5m^9}{6} - m^7 + m^5 - \frac{1}{2}m^3 + \frac{5}{66}m\right) dm = \frac{\alpha^{11}}{11}$$

4. Theorem 2

(a) If k is odd, then

$$\lim_{\alpha\to\infty} \frac{1}{\alpha^{k+2}}\left(S_{k,\alpha}(n)\right) = \frac{2}{(k+1)(k+2)} \quad (7)$$

(b) If k is even, then

$$\lim_{\alpha\to\infty} \frac{1}{\alpha^{k+1}}\left(S_{k,\alpha}(n)\right) = \frac{1}{k+1} \quad (8)$$

Proof:

(a) From the definition of $S_{k,\alpha}(n)$

$$S_{k,\alpha}(n) = \int_{-\alpha}^{\alpha} t_k(m)\, dm$$

$$= \int_{-\alpha}^{\alpha} \left(\frac{m^{k+1}}{k+1} + \frac{m^k}{2} + k\frac{m^{k-1}}{12} + C_{k-3}m^{k-3} + C_{k-5}m^{k-5} + \cdots + C_1 m\right) dm$$

(since $\int_{-\alpha}^{\alpha} f(x)\, dx = 0 \; for\ odd\ functions\ f$)

$$\lim_{\alpha\to\infty} \frac{1}{\alpha^{k+2}}\left(S_{k,\alpha}(n)\right) = \lim_{\alpha\to\infty} \frac{1}{\alpha^{k+2}} \int_{-\alpha}^{\alpha} \frac{m^{k+1}}{k+1}\, dm$$

$$= \lim_{\alpha\to\infty} \frac{1}{\alpha^{k+2}} \frac{1}{k+1} \left(2\, \frac{\alpha^{k+2}}{k+2}\right)$$

$$= \frac{1}{(k+1)(k+2)}$$

(b) If k is even, then $S_{k,\alpha}(n) = \frac{\alpha^{k+1}}{K+1}$ (from Theorem 1)

Therefore

$$\lim_{\alpha\to\infty} \frac{1}{\alpha^{k+1}}\left(S_{k,\alpha}(n)\right) = \lim_{\alpha\to\infty} \frac{1}{\alpha^{k+1}} \left(\frac{\alpha^{k+1}}{K+1}\right)$$

$$= \frac{1}{k+1}$$

This completes the proof.

Verification of Result (7)

The result obtained in (7) can be verified for odd k = 1,3,5,7,9

When k = 1

$$\lim_{\alpha\to\infty} \frac{1}{\alpha^{k+2}}\left(S_{1,\alpha}(n)\right) = \lim_{\alpha\to\infty} \frac{1}{\alpha^3}\left(\frac{\alpha^3}{3}\right) = \frac{1}{3} = \frac{2}{2(1+2)}$$

When k = 3

$$\lim_{\alpha\to\infty} \frac{1}{\alpha^{k+2}}\left(S_{3,\alpha}(n)\right) = \lim_{\alpha\to\infty} \frac{1}{\alpha^5}\left(\frac{\alpha^5}{10} + \frac{\alpha^3}{6}\right) = \frac{1}{10} = \frac{2}{(3+1)(3+2)}$$

When k = 5

$$\lim_{\alpha\to\infty} \frac{1}{\alpha^{k+2}}\left(S_{5,\alpha}(n)\right) = \lim_{\alpha\to\infty} \frac{1}{\alpha^7}\left(\frac{1}{3}\left(\frac{\alpha^7}{7} + \frac{\alpha^5}{2} - \frac{\alpha^3}{6}\right)\right) = \frac{1}{21} = \frac{2}{(5+1)(5+2)}$$

When k = 7

$$\lim_{\alpha\to\infty} \frac{1}{\alpha^{k+2}}\left(S_{7,\alpha}(n)\right) = \lim_{\alpha\to\infty} \frac{1}{\alpha^9}\left(\frac{1}{3}\left(\frac{\alpha^9}{12} + \frac{\alpha^7}{4} - \frac{7\alpha^5}{20} + \frac{\alpha^3}{6}\right)\right) = \frac{1}{36} = \frac{2}{(7+1)(7+2)}$$

When k = 9

$$\lim_{\alpha\to\infty} \frac{1}{\alpha^{k+2}}\left(S_{9,\alpha}(n)\right) = \lim_{\alpha\to\infty} \frac{1}{\alpha^{11}}\left(\frac{\alpha^{11}}{55} + \frac{\alpha^9}{6} - \frac{\alpha^7}{5} + \frac{\alpha^5}{5}\right) = \frac{1}{55} = \frac{2}{(9+1)(9+2)}$$

Verification of Result (8)

The result obtained in (8) can be verified for k = 2,4,6,8,10 when k = 2

$$\lim_{\alpha\to\infty} \frac{1}{\alpha^3}(S_{2,\alpha}(n)) = \lim_{\alpha\to\infty} \frac{1}{\alpha^3}\left(\frac{\alpha^3}{3}\right) = \frac{1}{3}$$

when k = 4

$$\lim_{\alpha\to\infty} \frac{1}{\alpha^5}((S_{4,\alpha}(n)) = \lim_{\alpha\to\infty} \frac{1}{\alpha^5}\left(\frac{\alpha^5}{5}\right) = \frac{1}{5}$$

when k =6

$$\lim_{\alpha\to\infty} \frac{1}{\alpha^7}((S_{6\,,\alpha}(n)) = \lim_{\alpha\to\infty} \frac{1}{\alpha^7}\left(\frac{\alpha^7}{7}\right) = \frac{1}{7}$$

when k = 8

$$\lim_{\alpha\to\infty} \frac{1}{\alpha^9}((S_{8,\alpha}(n)) = \lim_{\alpha\to\infty} \frac{1}{\alpha^9}\left(\frac{\alpha^9}{9}\right) = \frac{1}{9}$$

when k = 10

$$\lim_{\alpha\to\infty} \frac{1}{\alpha^{11}}((S_{10\,,\alpha}(n)) = \lim_{\alpha\to\infty} \frac{1}{\alpha^{11}}\left(\frac{\alpha^{11}}{11}\right) = \frac{1}{11}$$

Conclusion

In this paper, we have attempted to investigate methods of Summability theory by making a particular divergent series to convergent through a definite integral defined over a bounded interval. In doing so, we obtained three results as given in (4), (7) and (8). We notice that if the interval in (3) is unbounded, then $S_{k,\alpha}(n)$ is not summable. By considering suitable definite integrals and intervals, we can always generalize or create new results similar to those of obtained in this paper.

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